**EAC manifolds with structure group G₂**

Mahdi Kamandar* and Mohammad Mansori

Faculty member of Khatam-al-anbia air Defense University

**Corresponding Author:** Mahdi Kamandar

Received: 01 April, 2018                Accepted: 14 April, 2018                   Published: 30 April, 2018

**ABSTRACT**

In this paper we will consider the deformation theory of compact G₂-manifolds, where G = G₂. We will prove that the moduli space of torsion-free G₂-structures is a smooth manifold also proved smoothness of the moduli space on compact G₂-manifolds for any of the Ricci-at holonomy groups G₂ in a fairly uniform way. The arguments used here are geared to make it easier to generalise to the asymptotically cylindrical case in physics.

**Keywords:** EAC manifolds, G₂-manifolds, cylindrical.

©2018 GJSR Journal All rights reserved.

**INTRODUCTION**

A way to obtain irreducible compact G₂-manifolds is by gluing a pair of noncompact G₂-manifolds which are asymptotically cylindrical. A manifold is said to have cylindrical ends if it is homeomorphic to a cylinder outside a compact piece. An asymptotically cylindrical manifold is a Riemannian manifold with cylindrical ends for which the metric is asymptotic to a product metric on the cylindrical ends. Asymptotically cylindrical manifolds are easier to work with than arbitrary non-compact manifolds. Many analysis results for elliptic operators on compact manifolds can be generalised to statements about asymptotically translation-invariant elliptic operators acting on suitable spaces of sections on an asymptotically cylindrical manifold. In some arguments it is helpful to impose a stronger condition, requiring the manifold to be exponentially asymptotically cylindrical (EAC). Given a pair of EAC G₂-manifolds whose cylinders match one can form a generalised connected sum by truncating the cylinders after some large but finite length and gluing them together. If the neck length is sufficiently large then the EAC G₂-structures can be glued to form a torsion-free G₂-structure on the connected sum. This is a gluing construction for compact G₂-manifolds. Kovalev proves an EAC version of the Calabi conjecture to produce EAC Calabi-Yau 3-folds. By multiplying with circles reducible EAC G₂-manifolds are obtained, which can be glued to form irreducible compact G₂-manifolds different topological types from those constructed by Joyce.

**Definition 1.1.** Let X₀ be a compact manifold, and denote by t the R-coordinate on the cylinder X × R. Let M be a Riemannian manifold with HOL(M) ⊆ H and ρ a representation of H. The Lichnerowicz Laplacian on Eₚ is the formally self-adjoint operator

\[ \Delta_p = \nabla^* \nabla - 2 (D_p)^2 (R) : \mathcal{I}(E_p) \rightarrow \mathcal{I}(E_p) \]

where \( \nabla \) is the connection on Eₚ induced by the Levi-Civita connection on M.

The 43rd Annual Iranian Mathematics Conference, University of Tabriz
27 - 30 August 2012, Tabriz, Iran

**Definition 1.2.** A G₂-structure on X × R is cylindrical if it is translation-invariant and the associated metric is a product metric \( g_p = g_0 + dt^2 \)

**Definition 1.3.** A manifold M is said to have cylindrical ends if it is a union of two pieces \( M_0 \) and \( M_∞ \) with common boundary X, where \( M_0 \) is compact, and \( M_∞ \) is identified with \( X \times \mathbb{R}^* \) by a diffeomorphism (identifying \( \partial M_∞ \) with \( X \times \{0\} \) X is called the cross-section of M.
Definition 1.4. A tensor field or differential operator on $X \times \mathbb{R}$ is called translation invariant if it is invariant under the obvious $\mathbb{R}$-action on $X \times \mathbb{R}$.

Definition 1.5. An asymptotically cylindrical manifold is a Riemannian manifold with cylindrical ends for which the metric is asymptotic to a product metric on the cylindrical ends.

Definition 1.6. A metric $g$ on a manifold $M$ with cylindrical ends is said to be EAC if it is exponentially asymptotic to a product $g_\infty + dt^2$ metric on $X \times \mathbb{R}$. An EAC manifold is a manifold with cylindrical ends equipped with an EAC metric.

Proposition 1.7. Let $M$ be an EAC $G_2$-manifold with cross-section $X$. Then

$$H^2_b(X) = A^2_b \oplus E^2_b, H^6_b(X) = A^6_b \oplus E^6_b$$

and the sums are orthogonal. Furthermore

(i) $H^2_b(X) \to H^6_b(X), [\alpha] \to [\alpha]$ maps $A^2_b$ to $E^6_b$ and $E^2_b$ to $A^6_b$.

(ii) $H^4_b(X) \to H^6_b(X), [\alpha] \to [\alpha] \cup [\mathcal{Q}]$ maps $A^4$ to $A^6_b$ and $E^4$ to $E^6_b$.

(iii) $H^2(X) \to H^6_b(X), [\alpha] \to [\alpha] \cup \left[ \frac{1}{w^2} \right]$ maps $A^1$ to $A^2$ and $E^1$ to $E^2$.

Proof: (i) is obvious, since $\ast$ maps $A^m \mapsto E^{6-m}$.

$[\alpha] \to [\alpha] \cup [\mathcal{Q}]$ is a bijection $H^4_b(X) \to H^6_b(X)$. It maps $A^1$ into $A^4$ and $E^1$ into $E^4$. It follows that $A^4 \to A^6_b$ and $E^4 \to E^6_b$ are both surjective and that $H^6_b(X)$ splits as $A^6_b \oplus E^6_b$. $H^2_b(X)$ splits too by (i).

(iii) easily follows from (i) and (ii) in the same way.

Lemma 1.8. Let $M$ be a Ricci-at EAC manifold:

(i) If $M$ has a finite normal cover homeomorphic to a cylinder then $M$ or a double cover of $M$ is homeomorphic to a cylinder.

(ii) If $\pi_1(M)$ is infinite then $M$ has a finite cover $\tilde{M}$ with $b^1(\tilde{M}) > 0$.

Proof: (i) If $\tilde{M}$ is a finite normal cover of a manifold homoeomorphic to a cylinder then it is isometric to a product cylinder $Y \times \mathbb{R}$. $M$ is a quotient of $Y \times \mathbb{R}$ by a finite group of isometries. $M$ is homeomorphic to $Y \times \mathbb{R}$ if and only if it is homeomorphic to a product $Y \times \mathbb{R}$.

(iii) Let $G_0 \subseteq \pi_1(M)$ be a nilpotent subgroup of finite index. $G_0$ is soluble, so the derived series $G_i+1 = [G_i, G_i]$ reaches 1. Therefore there is a largest $i$ such that $G_i \subseteq \pi_1(M)$ has finite index. Let $\tilde{M}$ be the cover of $M$ corresponding to $G_i \subseteq \pi_1(M)$. $\frac{\tilde{G}_i}{G_{i+1}}$ is an infinite Abelian group, so has non-zero rank.

Theorem 1.9. Let $M$ be $M_\Sigma$ with its orientation reversed and $(\varphi_+, \varphi_-)$ a matching pair of $G_2$-structures. If $\varphi_+$ and $\varphi_-$ define the same metric then $M_\Sigma$ has a double cover isometric to a cylinder.

Proof. $\varphi_+$ is a torsion-free $G_2$-structure on $M_\Sigma$ which defines the same metric as $\varphi_-$. The matching condition for $\varphi_+$ and $\varphi_-$ implies that the parallel section is asymptotic to $\left[ \frac{\partial}{\partial t} \right]$. In other words either $M_\Sigma$ or a double cover of $M_\Sigma$ has a parallel vector field asymptotic to $\frac{\partial}{\partial t}$ now this is impossible for a manifold with a single end, so $M_\Sigma$ has a double cover which is isometric to a cylinder. Result 1.10. Let $M_\Sigma$ be the moduli space of torsion-free EAC $G_2$-structures on $M$ and $N$ the moduli space of Calabi-Yau structures on their common cross-section $X$. We can define a subset $M_\Sigma \subseteq M_\Sigma \times M$ consisting of pairs which have matching images in $N$.

While we can apply our understanding of $M_\Sigma$ and their relationship to $N$ to show that $M_\Sigma$ is a manifold, it is not an appropriate domain. The reason is that for a matching pair of points in the moduli spaces $M_\Sigma$, $M$ there is some ambiguity in how to glue them.

Corollary 1.11. Let $M$ be an asymptotically cylindrical manifold with non-negative Ricci curvature. Then the fundamental group $\pi_1(M)$ has a nilpotent subgroup of finite index. $M$ is homotopy equivalent to a compact manifold with boundary so $\pi_1(M)$ is finitely generated. Volume comparison arguments show that the volume of balls in the universal cover of $M$ grows polynomially and this can be used to deduce that $\pi_1(M)$ has polynomial growth.

2 Main Result

if $C = C_7 + C_{14}$ is a skew-symmetric tensor, then the evolution of the skew-symmetric tensor $P(C)$ under the ow equation:

$$\frac{\partial}{\partial t}(P(C))_{ij} = (P(\frac{\partial}{\partial t} C))_{ij} + 6\pi_7([h, C_{14}])_{ij} - 6\pi_4([h, C_7])_{ij} - 2\pi_7([X, C_{14}])_{ij} + 2\pi_4([X, C_7])_{ij}$$
Where $\pi_7$ and $\pi_{14}$ denote the projections onto $\Omega^2_7$ and $\Omega^2_{14}$ respectively.

Proof. we see that
\[ \frac{\partial}{\partial t} \left( C_{ab} g^{ap} g^{bq} \psi_{pqi} \right) \]
equals
\[ \left( \frac{\partial}{\partial t} C_{ab} \right) g^{ap} g^{bq} \psi_{pqi} + 2 C_{ab} \left( \frac{\partial}{\partial t} g^{ap} \right) g^{bq} \psi_{pqi} + C_{ab} g^{ap} g^{bq} \left( \frac{\partial}{\partial t} \psi_{pqi} \right) \]
\[ = \left( P \left( \frac{\partial}{\partial t} C \right) \right)_{ij} - 4 C_{ab} h^{ap} g^{bq} \psi_{pqi} + C_{ab} h^{ap} g^{bq} \left( h_p^i \psi_{lqi} + h_q^i \psi_{pil} \right) \]
\[ + C_{ab} g^{ap} g^{bq} \left( h_p^i \psi_{pil} + h_q^i \psi_{pqi} - X_p \psi_{qij} + X_q \psi_{pil} - X_i \psi_{paj} + X_j \psi_{pqi} \right) \]
\[ = \left( P \left( \frac{\partial}{\partial t} C \right) \right)_{ij} - 2 C_{ab} h^{ap} g^{bq} \psi_{pqi} + h_i^j \left( P(C) \right)_{ij} + \left( P(C) \right)_{ji} h_j^i \]
\[ + 2 \left( C_{ab} X^b g^{bq} \right) \psi_{pqi} - 6 (C_7)_i X_i + 6 (C_7)_i X_j \quad (1) \]

where we have used the skew-symmetry of $C$ and of $\varphi$ and relabeled indices to combine terms. The second term above can be written as
\[ -2 h_{at} g^{im} C_{mb} g^{ap} g^{bq} \psi_{pqi} = - (h_{at} g^{im} C_{mb} + C_{at} g^{im} h_{mb}) g^{ap} g^{bq} \psi_{pqi} \]
\[ = - \{ h, c \}_{ab} g^{ap} g^{bq} \psi_{pqi} = - P \{ h, c \}_{ij} = 4 (\pi_7 \{ h, c \})_{ij} - 2 (\pi_{14} \{ h, c \})_{ij} \]
\[ = 4 (\pi_7 \{ h, c_7 \})_{ij} + 2 (\pi_7 \{ h, c_{14} \})_{ij} - 2 (\pi_{14} \{ h, c_7 \})_{ij} - 2 (\pi_{14} \{ h, c_{14} \})_{ij} \]

Meanwhile the third and fourth terms of (1) become Combining these
\[ \{ h, P(C) \}_{ij} = \{ h, -4 C_7 + 2 c_{14} \}_{ij} \]
\[ = -4 (\pi_7 \{ h, c_7 \})_{ij} + 2 (\pi_7 \{ h, c_{14} \})_{ij} - 4 (\pi_{14} \{ h, c_7 \})_{ij} + 2 (\pi_{14} \{ h, c_{14} \})_{ij} \]

expressions, after some cancellation we see that
\[ \frac{\partial}{\partial t} \left( C_{ab} g^{ap} g^{bq} \psi_{pqi} \right) = \left( P \left( \frac{\partial}{\partial t} C \right) \right)_{ij} + 6 \pi_7 \{ h, c_{14} \}_{ij} - 6 \pi_{14} \{ h, c_7 \}_{ij} \]
\[ + 2 \left( C_{ab} X^b g^{bq} \right) \psi_{pqi} - 6 (C_7)_i X_i + 6 (C_7)_i X_j \quad (2) \]

Consider now the third to last term above:
\[ 2 \left( C_i (X_i) + C_i (X_i) \right) = 2 \left( - \frac{1}{2} [C_i, X_i] - \frac{3}{2} (C_7)_i X_i + \frac{3}{2} (C_7)_i X_i \right) \]
\[ = -[C_i, X_i] + 2 \left[ C_{14}, X_i \right] - 3 (C_7)_i X_i + 3 (C_7)_i X_i \]
Hence the final three terms of (2) are:
\[ = -[C_i, X_i] + 2 \left[ C_{14}, X_i \right] + 3 (C_7)_i X_i - 3 (C_7)_i X_i \]
\[ = -[C_i, X_i] + 2 \left[ C_{14}, X_i \right] - 3 \left( -\frac{1}{3} [C_i, X_i] + \frac{2}{3} (C_7 \times X)_{ij} \right) \]
\[ = -2 [C_i, X_i] + 2 \left[ C_{14}, X_i \right] + 2 (C_7 \times X)_{ij} \quad (3) \]

now by using (1) and (2) and (3) the result is prove.

REFERENCES